The Second Euler-Lagrange Equation
of Variational Calculus on Time Scales

Zbigniew Bartosiewicz, Natália Martins, and Delfim F. M. Torres

Abstract— **The fundamental problem of the calculus of variations on time scales concerns the minimization of a delta-integral over all trajectories satisfying given boundary conditions. In this paper we prove the second Euler-Lagrange necessary optimality condition for optimal trajectories of variational problems on time scales. As an example of application of the main result, we give an alternative and simpler proof to the Noether theorem on time scales recently obtained in [J. Math. Anal. Appl. 342 (2008), no. 2, 1220–1226].**

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I. INTRODUCTION

The calculus on time scales is a recent field, introduced by Bernd Aulbach and Stefan Hilger in 1988 [5], that unifies and extends difference and differential equations into a single theory [14]. A time scale is a model of time, and the new theory has found important applications in several fields that require simultaneous modeling of discrete and continuous data, in particular in the calculus of variations [2], [3], [4], [13], [19], [20], [24], [28], [29], [30], control theory [7], [8], [9], [10], [18], [31], and optimal control [25], [32], [39]. Other areas of application include engineering, biology, economics, finance, and physics [1], [14].

The present work is dedicated to the study of problems of calculus of variations on an arbitrary time scale $\mathbb{T}$. As particular cases, one gets the standard calculus of variations [16], [38] by choosing $\mathbb{T} = \mathbb{R}$; the discrete-time calculus of variations [26], [34] by choosing $\mathbb{T} = \mathbb{Z}$; and the q-calculus of variations [6] by choosing $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k | k \in \mathbb{N}_0\}$, $q > 1$. In Section II we briefly present the necessary notions and results of time scales, delta derivatives, and delta integrals.

Let $\mathbb{T}$ be a given time scale with at least three points, $n \in \mathbb{N}$, and $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be of class C^1. Suppose that

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$a, b \in \mathbb{T}$ and $a < b$. We consider the following optimization problem on $\mathbb{T}$:

$$I[q] = \int_a^b L(t, q^\sigma(t), q^\Delta(t)) \Delta t \longrightarrow \min_{q \in \mathscr{D}}, \quad (1)$$

where

$$\mathscr{D} = \{q \mid q : [a,b] \cap \mathbb{T} \to \mathbb{R}^n, \ q \in \mathrm{C}^1_{rd}, \ q(a) = q_a, \ q(b) = q_b\}$$

for some $q_a, q_b \in \mathbb{R}^n$, and where σ is the forward jump operator and q^Δ is the delta-derivative of q with respect to $\mathbb{T}$. For $\mathbb{T} = \mathbb{R}$ we get the classical fundamental problem of the calculus of variations, which concerns the minimization of an integral

$$I[q] = \int_a^b L(t, q(t), \dot{q}(t)) dt \quad (2)$$

over all trajectories $q \in \mathrm{C}^1$ satisfying given boundary conditions $q(a) = q_a$ and $q(b) = q_b$. Several classical results on the calculus of variations are now available to the more general context of time scales: (first) Euler-Lagrange equations [4], [13], [24]; necessary optimality conditions for isoperimetric problems [3], [28] and for problems with higher-order derivatives [20], [30]; the Weierstrass necessary condition [29]; and Noether's symmetry theorem [11]. In this paper we prove a new result for the problem of the calculus of variations on time scales: we obtain in Section III a time scale version of the classical *second Euler-Lagrange equation* [37], also known in the literature as the DuBois-Reymond necessary optimality condition [15].

The classical second Euler-Lagrange equation asserts that if q is a minimizer of (2), then

$$\frac{d}{dt} \left[-L(t, q(t), \dot{q}(t)) + \partial_3 L(t, q(t), \dot{q}(t)) \cdot \dot{q}(t) \right]$$
$$= -\partial_1 L(t, q(t), \dot{q}(t)), \quad (3)$$

where $\partial_i L$, $i = 1, 2, 3$, denotes the partial derivative of $L(\cdot, \cdot, \cdot)$ with respect to its i-th argument. In the autonomous case, when the Lagrangian L does not depend on the time variable t, the second Euler-Lagrange condition (3) is nothing more than the second Erdmann necessary optimality condition:

$$-L(q(t), \dot{q}(t)) + \partial_3 L(q(t), \dot{q}(t)) \cdot \dot{q}(t) = const \quad (4)$$

along all the extremals of the problem, which in mechanics corresponds to the most famous conservation law—conservation of energy. For a survey of the classical optimality conditions we refer the reader to [17, Ch. 2]. Here we just recall that (3) is one of the cornerstone results of the calculus of variations and optimal control [35]: it has

been used, for example, to prove existence, regularity of minimizers, conservation laws, and to explain the Lavrentiev phenomena.

Main result of the paper gives an extension of (3) to an arbitrary time scale (cf. Theorem 5): if q is a solution of problem (1), then

$$\frac{\Delta}{\Delta t}\Big[-L(t,q^\sigma(t),q^\Delta(t))+\partial_3 L(t,q^\sigma(t),q^\Delta(t))q^\Delta(t)$$
$$+\partial_1 L(t,q^\sigma(t),q^\Delta(t))\mu(t)\Big]=-\partial_1 L(t,q^\sigma(t),q^\Delta(t)). \quad (5)$$

As an application, we show in Section IV how one can use the new second Euler-Lagrange equation (5) to prove the Noether's principle on time scales: the invariance of functional (1) with respect to a one-parameter family of transformations implies the existence of a conserved quantity along the time scale Euler-Lagrange extremals (Theorem 6). When problem (1) is autonomous one has invariance with respect to time translations and the corresponding Noether's conservation law gives an extension of the second Erdmann equation (4) to time scales:

$$-L(q^\sigma(t),q^\Delta(t))+\partial_3 L(q^\sigma(t),q^\Delta(t))\cdot q^\Delta(t)=const. \quad (6)$$

In classical mechanics (4) gives conservation of energy. The conservation law (6) tells us that an analogous result remains valid on an arbitrary time scale. However, the role of the classical Hamiltonian $H(t,u,v)=-L(t,u,v)+\partial_3 L(t,u,v)v$ in (3) is substituted by a time-scale Hamiltonian

$$\mathscr{H}(t,u,v)=-L(t,u,v)+\partial_3 L(t,u,v)v+\partial_1 L(t,u,v)\mu(t)$$

in (5), i.e., $\mathscr{H}(t,u,v)=H(t,u,v)+\partial_1 L(t,u,v)\mu(t)$, with the new term $\partial_1 L(t,q^\sigma(t),q^\Delta(t))\mu(t)$ on the left-hand side of (5) depending on the graininess of the time scale.

II. PRELIMINARIES ON TIME SCALES

For a general introduction to the calculus on time scales we refer the reader to the book [14]. Here we only give those notions and results needed in the sequel. As usual, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ denote, respectively, the set of real, integer, and natural numbers.

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. Besides standard cases of $\mathbb{R}$ (continuous time) and $\mathbb{Z}$ (discrete time), many different models of time are used. For each time scale $\mathbb{T}$ the following operators are used:

- the *forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$, $\sigma(t):=\inf\{s\in\mathbb{T}:s>t\}$ for $t\neq\sup\mathbb{T}$ and $\sigma(\sup\mathbb{T})=\sup\mathbb{T}$ if $\sup\mathbb{T}<+\infty$;
- the *backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$, $\rho(t):=\sup\{s\in\mathbb{T}:s<t\}$ for $t\neq\inf\mathbb{T}$ and $\rho(\inf\mathbb{T})=\inf\mathbb{T}$ if $\inf\mathbb{T}>-\infty$;
- the *forward graininess function* $\mu:\mathbb{T}\to[0,\infty[$, $\mu(t):=\sigma(t)-t$.

Example 1: If $\mathbb{T}=\mathbb{R}$, then for any $t\in\mathbb{R}$, $\sigma(t)=t=\rho(t)$ and $\mu(t)\equiv 0$. If $\mathbb{T}=\mathbb{Z}$, then for every $t\in\mathbb{Z}$, $\sigma(t)=t+1$, $\rho(t)=t-1$ and $\mu(t)\equiv 1$.

A point $t\in\mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense* or *left-scattered* if $\sigma(t)=t$, $\sigma(t)>t$, $\rho(t)=t$, or $\rho(t)<t$, respectively. We say that t is *isolated* if $\rho(t)<t<\sigma(t)$, that t is *dense* if $\rho(t)=t=\sigma(t)$.

If $\sup\mathbb{T}$ is finite and left-scattered, we define

$$\mathbb{T}^\kappa:=\mathbb{T}\setminus\{\sup\mathbb{T}\}.$$

Otherwise, $\mathbb{T}^\kappa:=\mathbb{T}$.

Definition 1: Let $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^\kappa$. The *delta derivative* of f at t is the real number $f^\Delta(t)$ with the property that given any $\varepsilon>0$ there is a neighborhood U of t such that

$$|(f(\sigma(t))-f(s))-f^\Delta(t)(\sigma(t)-s)|\leq\varepsilon|\sigma(t)-s|$$

for all $s\in U$. We say that f is *delta differentiable* on $\mathbb{T}$ provided $f^\Delta(t)$ exists for all $t\in\mathbb{T}^\kappa$.

We shall often denote $f^\Delta(t)$ by $\frac{\Delta}{\Delta t}f(t)$ if f is a composition of other functions. The delta derivative of a function $f:\mathbb{T}\to\mathbb{R}^n$ ($n\in\mathbb{N}$) is a vector whose components are delta derivatives of the components of f. For $f:\mathbb{T}\to X$, where X is an arbitrary set, we define $f^\sigma:=f\circ\sigma$.

For delta differentiable f and g, the next formulas hold:

$$f^\sigma(t)=f(t)+\mu(t)f^\Delta(t),$$
$$(fg)^\Delta(t)=f^\Delta(t)g^\sigma(t)+f(t)g^\Delta(t)$$
$$=f^\Delta(t)g(t)+f^\sigma(t)g^\Delta(t).$$

Remark 1: If $\mathbb{T}=\mathbb{R}$, then $f:\mathbb{R}\to\mathbb{R}$ is delta differentiable at $t\in\mathbb{R}$ if and only if f is differentiable in the ordinary sense at t. Then, $f^\Delta(t)=\frac{d}{dt}f(t)$. If $\mathbb{T}=\mathbb{Z}$, then $f:\mathbb{Z}\to\mathbb{R}$ is always delta differentiable at every $t\in\mathbb{Z}$ with $f^\Delta(t)=f(t+1)-f(t)$.

Let $a,b\in\mathbb{T}$, $a<b$. We define the interval $[a,b]_\mathbb{T}$ in $\mathbb{T}$ by

$$[a,b]_\mathbb{T}:=\{t\in\mathbb{T}:a\leq t\leq b\}.$$

Open intervals and half-open intervals in $\mathbb{T}$ are defined accordingly.

Theorem 1 (Corollary 2.9 of [23]): Let $f:[a,b]_\mathbb{T}\to\mathbb{R}$ be a continuous function that has a delta derivative at each point of $[a,b]_\mathbb{T}^\kappa$. Then f is increasing, decreasing, non-decreasing, and non-increasing on $[a,b]_\mathbb{T}$ if $f^\Delta(t)>0$, $f^\Delta(t)<0$, $f^\Delta(t)\geq 0$ and $f^\Delta(t)\leq 0$ for all $t\in[a,b]_\mathbb{T}^\kappa$, respectively.

Definition 2: A function $F:\mathbb{T}\to\mathbb{R}$ is called a *delta antiderivative* of $f:\mathbb{T}\to\mathbb{R}$ provided

$$F^\Delta(t)=f(t), \qquad \forall t\in\mathbb{T}^\kappa.$$

In this case we define the *delta integral* of f from a to b ($a,b\in\mathbb{T}$) by

$$\int_a^b f(t)\Delta t:=F(b)-F(a).$$

Example 2: If $\mathbb{T}=\mathbb{R}$, then

$$\int_a^b f(t)\Delta t=\int_a^b f(t)dt,$$

where the integral on the right hand side is the usual Riemann integral. If $\mathbb{T}=h\mathbb{Z}$, where $h>0$, then

$$\int_a^b f(t)\Delta t=\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} h\cdot f(kh),$$

for $a<b$.

In order to present a class of functions that possess a delta antiderivative, the following definition is introduced:

Definition 3: A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at the right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}^n$ is *rd-continuous* if all its components are rd-continuous.

We remark that a rd-continuous function defined on a compact interval, with real values, is bounded. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}^n$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R}^n)$, or simply by C_{rd}. Similarly, $C^1_{rd}(\mathbb{T}, \mathbb{R}^n)$ and C^1_{rd} will denote the set of functions from C_{rd} whose delta derivative belongs to C_{rd}.

Theorem 2 (Theorem 1.74 of [14]): Every rd-continuous function has a delta antiderivative. In particular, if $a \in \mathbb{T}$, then the function F defined by

$$F(t) = \int_a^t f(\tau) \Delta \tau, \quad t \in \mathbb{T},$$

is a delta antiderivative of f.

The following results will be very useful in the proof of our main result (Theorem 5).

Theorem 3 (Theorems 1.93, 1.97, and 1.98 of [14]): Assume that $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale.

1) (Chain rule) Let $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $v^\Delta(t)$ and $\omega^{\widetilde{\Delta}}(v(t))$ exist for all $t \in \mathbb{T}^\kappa$, then

$$(\omega \circ v)^\Delta = (\omega^{\widetilde{\Delta}} \circ v) v^\Delta.$$

2) (Derivative of the inverse) The relation

$$(v^{-1})^{\widetilde{\Delta}}(v(t)) = \frac{1}{v^\Delta(t)}$$

holds at points $t \in \mathbb{T}^\kappa$ where $v^\Delta(t) \neq 0$.

3) (Substitution in the integral) If $f : \widetilde{\mathbb{T}} \to \mathbb{R}$ is a C_{rd} function and v is a C^1_{rd} function, then for $a, b \in \mathbb{T}$,

$$\int_a^b f(v(t)) v^\Delta(t) \Delta t = \int_{v(a)}^{v(b)} f(s) \widetilde{\Delta} s.$$

Definition 4: We say that $y_* \in C^1_{rd}([a,b]_\mathbb{T}, \mathbb{R}^n)$ is a local minimizer for problem (1) if there exists $\delta > 0$ such that

$$I[y_*] \leq I[y]$$

for all $y \in C^1_{rd}([a,b]_\mathbb{T}, \mathbb{R}^n)$ satisfying the boundary conditions $q(a) = q_a$, $q(b) = q_b$, and

$$\| y - y_* \| := \sup_{t \in [a,b]_\mathbb{T}^\kappa} | y^\sigma(t) - y_*^\sigma(t) | + \sup_{t \in [a,b]_\mathbb{T}^\kappa} | y^\Delta(t) - y_*^\Delta(t) | < \delta,$$

where $|\cdot|$ denotes a norm in $\mathbb{R}^n$.

We recall now the (first) Euler-Lagrange equation as presented in [13]. As in the introduction, we use $\partial_i L$ to denote the partial derivative of L with respect to the i-th variable (or group of variables).

Theorem 4 (Theorem 4.2 of [13]): If q is a local minimizer of (1), then q satisfies the following Euler-Lagrange equation:

$$\frac{\Delta}{\Delta t} \partial_3 L\left(t, q^\sigma(t), q^\Delta(t)\right) = \partial_2 L\left(t, q^\sigma(t), q^\Delta(t)\right), \quad t \in [a,b]_\mathbb{T}^\kappa. \tag{7}$$

III. MAIN RESULTS

The following theorem presents a generalization to time scales of the second Euler-Lagrange equation [37] (also known as the DuBois-Reymond equation [15]).

Theorem 5: (the second Euler-Lagrange equation on time scales): If $q \in \mathscr{D}$ is a local minimizer of problem (1), then q satisfies the equation

$$\frac{\Delta}{\Delta t} \mathscr{H}(t, q^\sigma(t), q^\Delta(t)) = -\partial_1 L(t, q^\sigma(t), q^\Delta(t)) \tag{8}$$

for all $t \in [a,b]_\mathbb{T}^\kappa$, where

$$\mathscr{H}(t, u, v) = -L(t, u, v) + \partial_3 L(t, u, v) v + \partial_1 L(t, u, v) \mu(t),$$

$t \in \mathbb{T}$ and $u, v \in \mathbb{R}^n$.

Proof: Let $q_0 \in \mathscr{D}$ be a local minimizer of functional I in (1). We will prove that there exists $c \in \mathbb{R}^n$, $c \neq 0$, that satisfies the condition

$$1 - c^T q_0^\Delta(t) > 0, \quad \forall t \in [a,b]_\mathbb{T}^\kappa. \tag{9}$$

If $q_0^\Delta = 0$, then any $c \in \mathbb{R}^n$ satisfies condition (9). Suppose now that $q_0^\Delta \neq 0$. Then there exists some $i = 1, 2, \ldots, n$ such that $q_{0,i}^\Delta \neq 0$ where we suppose that $q_0 = (q_{0,1}, q_{0,2}, \ldots, q_{0,n})$. Since $q_{0,i}^\Delta$ is bounded on $[a,b]_\mathbb{T}^\kappa$, then there exist $m, M \in \mathbb{R}$ such that

$$m \leq q_{0,i}^\Delta(t) \leq M, \quad \forall t \in [a,b]_\mathbb{T}^\kappa.$$

Let $c := (c_1, c_2, \ldots, c_n)$ where $c_j = 0$ if $j \neq i$. If $M > 0$ we can choose c_i such that $0 < c_i < \frac{1}{M}$. If $M \leq 0$ we can choose c_i such that $\frac{1}{m} < c_i < 0$.

The map $S : [a,b]_\mathbb{T} \to \mathbb{R}$ defined by

$$S(t) = t - c^T q_0(t)$$

is delta differentiable with $S^\Delta(t) = 1 - c^T q_0^\Delta(t)$ and, by Theorem 1, S is strictly increasing on $[a,b]_\mathbb{T}$. Note that $\widetilde{\mathbb{T}} = S([a,b]_\mathbb{T})$ is a new time scale (because S is continuous and $[a,b]_\mathbb{T}$ is closed). By $\widetilde{\sigma}$ we denote the forward jump operator and by $\widetilde{\Delta}$ we denote the delta derivative on $\widetilde{\mathbb{T}}$. Let $\tau = S(t)$ and define $\eta_0(\tau) := q_0(S^{-1}(\tau))$ for $\tau \in \widetilde{\mathbb{T}}$. Note that

$$t = S^{-1}(\tau) = \tau + c^T \eta_0(\tau),$$

and

$$\eta_0(\tau) = q_0(\tau + c^T \eta_0(\tau)). \tag{10}$$

By the chain rule and from (10),

$$\eta_0^{\widetilde{\Delta}}(\tau) = q_0^\Delta(\tau + c^T \eta_0(\tau))(1 + c^T \eta_0^{\widetilde{\Delta}}(\tau)),$$

which gives

$$q_0^\Delta(t) = \frac{\eta_0^{\widetilde{\Delta}}(\tau)}{1 + c^T \eta_0^{\widetilde{\Delta}}(\tau)}. \tag{11}$$

By the derivative of the inverse applied to S we can conclude that

$$\frac{1}{1 + c^T \eta_0^{\widetilde{\Delta}}(\tau)} = 1 - c^T q_0^\Delta(t). \tag{12}$$

Note that, since $\widetilde{\sigma} \circ S = S \circ \sigma$, then

$$\begin{aligned}
\widetilde{\mu}(\tau) &= \widetilde{\sigma}(\tau) - \tau = \widetilde{\sigma}(S(t)) - S(t) \\
&= S^\sigma(t) - S(t) = \mu(t) S^\Delta(t) \\
&= \mu(t)(1 - c^T q_0^\Delta(t))
\end{aligned} \tag{13}$$

and

$$\eta_0(\tilde{\sigma}(\tau)) = \eta_0(\tilde{\sigma}(S(t)))$$
$$= \eta_0(S \circ \sigma)(t)) = q_0(S^{-1}(S \circ \sigma)(t)) \quad (14)$$
$$= q_0(\sigma(t)).$$

From (11), (14), and the substitution in the integral,

$$I[q_0] = \int_a^b L(t, q_0^\sigma(t), q_0^\Delta(t))\Delta t$$
$$= \int_\alpha^\beta \tilde{L}(\tau, \eta_0^{\tilde{\sigma}}(\tau), \eta_0^{\tilde{\Delta}}(\tau))\tilde{\Delta}\tau =: \tilde{I}[\eta_0],$$

where

$$\tilde{L}(\tau, v, \zeta) = L\left(\tau + c^T v - c^T\tilde{\mu}(\tau)\zeta, v, \frac{\zeta}{1 + c^T\zeta}\right)(1 + c^T\zeta),$$
(15)

for $\tau \in \tilde{\mathbb{T}}$, $v, \zeta \in \mathbb{R}^n$, $1 + c^T\zeta > 0$, $\alpha = S(a)$ and $\beta = S(b)$. Let

$$\mathscr{E} = \{\eta \mid \eta : \tilde{\mathbb{T}} \to \mathbb{R}^n, \ \eta \in C_{rd}^1,$$
$$\eta(\alpha) = \eta_0(\alpha), \ \eta(\beta) = \eta_0(\beta),$$
$$1 + c^T\eta^{\tilde{\Delta}}(\tau) > 0 \text{ for } \tau \in \tilde{\mathbb{T}}\}.$$

We remark that c was chosen so small that the constraint $1 + c^T\eta^{\tilde{\Delta}}(\tau) > 0$ is always satisfied for any function η in the "nearby" of η_0. Since q_0 is by assumption a local minimizer of I in $\mathscr{D}$, it follows that η_0 is a local minimizer of $\tilde{I}$ in $\mathscr{E}$, so it satisfies the Euler-Lagrange equation (in integral form)

$$\partial_3\tilde{L}(\tau, \eta_0^{\tilde{\sigma}}(\tau), \eta_0^{\tilde{\Delta}}(\tau)) = \int_\alpha^\tau \partial_2\tilde{L}(s, \eta_0^{\tilde{\sigma}}(s), \eta_0^{\tilde{\Delta}}(s))\tilde{\Delta}s + C_1,$$
(16)

where C_1 is a constant vector. Differentiating (15) we obtain

$$\partial_2\tilde{L}(\tau, v, \zeta)$$
$$= \partial_1 L\left(\tau + c^T v - c^T\tilde{\mu}(\tau)\zeta, v, \frac{\zeta}{1 + c^T\zeta}\right)(1 + c^T\zeta)c^T$$
$$+ \partial_2 L\left(\tau + c^T v - c^T\tilde{\mu}(\tau)\zeta, v, \frac{\zeta}{1 + c^T\zeta}\right)(1 + c^T\zeta)$$

and

$$\partial_3\tilde{L}(\tau, v, \zeta)$$
$$= L\left(\tau + c^T v - c^T\tilde{\mu}(\tau)\zeta, v, \frac{\zeta}{1 + c^T\zeta}\right)c^T$$
$$+ \partial_3 L\left(\tau + c^T v - c^T\tilde{\mu}(\tau)\zeta, v, \frac{\zeta}{1 + c^T\zeta}\right)(1 + c^T\zeta)^{-1}$$
$$- \partial_1 L\left(\tau + c^T v - c^T\tilde{\mu}(\tau)\zeta, v, \frac{\zeta}{1 + c^T\zeta}\right)c^T\tilde{\mu}(\tau)(1 + c^T\zeta).$$

Using (11), (12), (13) and (14) we obtain

$$\partial_3\tilde{L}\left(\tau, \eta_0^{\tilde{\sigma}}(\tau), \eta_0^{\tilde{\Delta}}(\tau)\right)$$
$$= L\left(\tau + c^T\eta_0^{\tilde{\sigma}}(\tau) - c^T\eta_0^{\tilde{\Delta}}(\tau)\tilde{\mu}(\tau), \eta_0^{\tilde{\sigma}}(\tau), \frac{\eta_0^{\tilde{\Delta}}(\tau)}{1 + c^T\eta_0^{\tilde{\Delta}}(\tau)}\right)c^T$$
$$+ \partial_3 L\left(\tau + c^T\eta_0^{\tilde{\sigma}}(\tau) - c^T\eta_0^{\tilde{\Delta}}(\tau)\tilde{\mu}(\tau), \eta_0^{\tilde{\sigma}}(\tau), \frac{\eta_0^{\tilde{\Delta}}(\tau)}{1 + c^T\eta_0^{\tilde{\Delta}}(\tau)}\right)$$
$$\cdot \left(1 + c^T\eta_0^{\tilde{\Delta}}(\tau)\right)^{-1} - c^T\tilde{\mu}(\tau)\left(1 + c^T\eta_0^{\tilde{\Delta}}(t)\right)$$
$$\cdot \partial_1 L\left(\tau + c^T\eta_0^{\tilde{\sigma}}(\tau) - c^T\eta_0^{\tilde{\Delta}}(\tau)\tilde{\mu}(\tau), \eta_0^{\tilde{\sigma}}(\tau), \frac{\eta_0^{\tilde{\Delta}}(\tau)}{1 + c^T\eta_0^{\tilde{\Delta}}(\tau)}\right)$$
$$= c^T L\left(t, q_0^\sigma(t), q_0^\Delta(t)\right) + \partial_3 L\left(t, q_0^\sigma(t), q_0^\Delta(t)\right)\left(1 - c^T q_0^\Delta(t)\right)$$
$$- c^T\mu(t)\left(1 - c^T q_0^\Delta(t)\right)\left(1 + c^T\eta_0^{\tilde{\Delta}}(t)\right)\partial_1 L\left(t, q_0^\sigma(t), q_0^\Delta(t)\right)$$
$$= c^T L\left(t, q_0^\sigma(t), q_0^\Delta(t)\right) + \partial_3 L\left(t, q_0^\sigma(t), q_0^\Delta(t)\right)\left(1 - c^T q_0^\Delta(t)\right)$$
$$- c^T\mu(t)\partial_1 L\left(t, q_0^\sigma(t), q_0^\Delta(t)\right).$$

Note that

$$\int_\alpha^\tau \partial_2\tilde{L}\left(s, \eta_0^{\tilde{\sigma}}(s), \eta_0^{\tilde{\Delta}}(s)\right)\tilde{\Delta}s + C_1$$
$$= \int_\alpha^\tau c^T\partial_1 L\left(S^{-1}(s), q_0^\sigma(S^{-1}(s)), q_0^\Delta(S^{-1}(s))\right)(S^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta}s$$
$$+ \int_\alpha^\tau \partial_2 L\left(S^{-1}(s), q_0^\sigma(S^{-1}(s)), q_0^\Delta(S^{-1}(s))\right)(S^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta}s + C_1$$
$$= \int_a^t c^T\partial_1 L\left(s, q_0^\sigma(s), q_0^\Delta(s)\right)\Delta s$$
$$+ \int_a^t \partial_2 L\left(s, q_0^\sigma(s), q_0^\Delta(s)\right)\Delta s + C_1.$$

Hence, by the Euler-Lagrange equation (16), we may conclude that

$$c^T L\left(t, q_0^\sigma(t), q_0^\Delta(t)\right) + \partial_3 L\left(t, q_0^\sigma(t), q_0^\Delta(t)\right)\left(1 - c^T q_0^\Delta(t)\right)$$
$$- c^T\mu(t)\partial_1 L\left(t, q_0^\sigma(t), q_0^\Delta(t)\right)$$
$$= \int_a^t c^T\partial_1 L\left(s, q_0^\sigma(s), q_0^\Delta(s)\right)\Delta s$$
$$+ \int_a^t \partial_2 L\left(s, q_0^\sigma(s), q_0^\Delta(s)\right)\Delta s + C_1.$$

The last equality may be rewritten as

$$c^T\left[L\left(t, q_0^\sigma(t), q_0^\Delta(t)\right) - \partial_3 L\left(t, q_0^\sigma(t), q_0^\Delta(t)\right)q_0^\Delta(t)\right.$$
$$\left. - \int_a^t \partial_1 L\left(s, q_0^\sigma(s), q_0^\Delta(s)\right)\Delta s - \partial_1 L\left(t, q_0^\sigma(t), q_0^\Delta(t)\right)\mu(t)\right]$$
$$= -\left[\partial_3 L\left(t, q_0^\sigma(t), q_0^\Delta(t)\right) - \int_a^t \partial_2 L\left(s, q_0^\sigma(s), q_0^\Delta(s)\right)\Delta s - C_1\right].$$

Using the Euler-Lagrange equation for q_0 we arrive at the intended statement. ∎

If $\mathbb{T} = \mathbb{R}$, then the equation (8) simplifies due to the fact that $\mu = 0$, and we obtain the classical second Euler-Lagrange equation (cf., e.g., [37]):

Corollary 1 (the second Euler-Lagrange equation): If q is a local minimizer of the classical functional (2) of the

calculus of variations, then

$$\frac{d}{dt}\left[-L(t,q(t),\dot{q}(t))+\partial_3 L(t,q(t),\dot{q}(t))\cdot\dot{q}(t))\right]$$
$$=-\partial_1 L(t,q(t),\dot{q}(t))$$

holds for all $t\in[a,b]$.

In the autonomous case, Theorem 5 gives an extension of the classical second Erdmann condition (4):

Corollary 2 (the second Erdmann condition on time scales): If $q\in\mathscr{D}$ is a local minimizer of the problem

$$I[q]=\int_a^b L(q^\sigma(t),q^\Delta(t))\Delta t\longrightarrow\min_{q\in\mathscr{D}},$$

then q satisfies equation (6) for all $t\in[a,b]_{\mathbb{T}}^\kappa$.

Example 3: Let $\mathbb{T}$ be a time scale with $a,b\in\mathbb{T}$, $a<b$. Consider problem (1) with $n=1$ and a Lagrangian L given by $L(t,q^\sigma,q^\Delta)=(q^\Delta)^2$. The second Euler-Lagrange equation (8) for this problem is

$$\frac{\Delta}{\Delta t}((q^\Delta(t))^2)=0,$$

and the extremal is $q(t)=ct+k$ with

$$c=\frac{q_b-q_a}{b-a},\quad k=\frac{bq_a-aq_b}{b-a}.$$

Example 4: Let $\mathbb{T}=\{0,\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{3}{4},\frac{7}{8},1\}$ and consider the following problem on $\mathbb{T}$:

$$I[q]=\int_0^1\left[(q^\Delta(t))^2-1\right]^2\Delta t\longrightarrow\min,$$
$$q(0)=0,\quad q(1)=0,$$
$$q\in C^1_{rd}(\mathbb{T};\mathbb{R}).$$

The Euler-Lagrange equation (7) takes the form

$$q^\Delta(t)\left[(q^\Delta(t))^2-1\right]=\text{const}\qquad(17)$$

while the second Euler-Lagrange equation (8) asserts that

$$\left[(q^\Delta(t))^2-1\right]\left[1+3(q^\Delta(t))^2\right]=\text{const}.\qquad(18)$$

Let $\tilde{q}(t)=0$ for all $t\in\mathbb{T}\setminus\{\frac{1}{8},\frac{7}{8}\}$, and $\tilde{q}\left(\frac{1}{8}\right)=\tilde{q}\left(\frac{7}{8}\right)=\frac{1}{8}$. One has $\tilde{q}^\Delta(0)=\tilde{q}^\Delta\left(\frac{3}{4}\right)=1$, $\tilde{q}^\Delta\left(\frac{1}{8}\right)=\tilde{q}^\Delta\left(\frac{7}{8}\right)=-1$, and $\tilde{q}^\Delta\left(\frac{i}{8}\right)=0$, $i=2,3,4,5$. We see that $\tilde{q}$ is an extremal, i.e., it satisfies the Euler-Lagrange equation (17). However $\tilde{q}$ cannot be a solution to the problem since it does not satisfy the second Euler-Lagrange equation (18). In fact, any function q satisfying $q^\Delta(t)\in\{-1,0,1\}$, $t\in\mathbb{T}^\kappa$, is an Euler-Lagrange extremal. Among them, only $q^\Delta(t)=0$ for all $t\in\mathbb{T}^\kappa$ and those with $q^\Delta(t)=\pm1$ satisfy our condition (18). This example shows a problem for which the Euler-Lagrange equation gives several candidates which are not the solution to the problem, while our second Euler-Lagrange equation gives a smaller set of candidates. Moreover, the candidates obtained from our condition lead us directly to the explicit solution of the problem. Indeed, the null function and any function q with $q(0)=q(1)=0$ and $q^\Delta(t)=\pm1$, $t\in\mathbb{T}^\kappa$, gives $I[q]=0$. They are minimizers because $I[q]\geq0$ for any function $q\in C^1_{rd}$.

IV. AN APPLICATION: NOETHER'S THEOREM

Let $U=\{q\mid q:[a,b]_{\mathbb{T}}\to\mathbb{R}^n,\ q\in C^1_{rd}\}$, and consider a one-parameter family of infinitesimal transformations

$$\begin{cases}\bar{t}=T(t,q,\varepsilon)=t+\varepsilon\tau(t,q)+o(\varepsilon),\\\bar{q}=Q(t,q,\varepsilon)=q+\varepsilon\xi(t,q)+o(\varepsilon),\end{cases}\qquad(19)$$

where $\varepsilon\in\mathbb{R}$, $\tau:[a,b]_{\mathbb{T}}\times\mathbb{R}^n\to\mathbb{R}$, and $\xi:[a,b]_{\mathbb{T}}\times\mathbb{R}^n\to\mathbb{R}$ are delta differentiable functions.

We assume that for every $q\in U$ and every ε, the map $[a,b]\ni t\mapsto\alpha(t):=T(t,q(t),\varepsilon)\in\mathbb{R}$ is a strictly increasing C^1_{rd} function and its image is again a time scale with the forward shift operator $\bar{\sigma}$ and the delta derivative $\bar{\Delta}$. We recall that the following holds:

$$\bar{\sigma}\circ\alpha=\alpha\circ\sigma.$$

Definition 5: Functional I in (1) is said to be invariant on U under the family of transformations (19) if

$$\frac{d}{d\varepsilon}\left\{L\left(T(t,q(t),\varepsilon),Q^\sigma(t,q(t),\varepsilon),\frac{Q^\Delta}{T^\Delta}\right)T^\Delta\right\}\bigg|_{\varepsilon=0}=0,$$

where, for simplicity of notation, we omit the arguments of functions T^Δ and Q^Δ: $T^\Delta=T^\Delta(t,q(t),\varepsilon)$, $Q^\Delta=Q^\Delta(t,q(t),\varepsilon)$.

Remark 2: Note that the invariance notion presented in [11, Definition 5] implies Definition 5. Indeed, for any subinterval $[t_a,t_b]_{\mathbb{T}}\subseteq[a,b]_{\mathbb{T}}$, any $q\in U$, and any ε, one has

$$\int_{t_a}^{t_b}L(t,q^\sigma(t),q^\Delta(t))\Delta t$$
$$=\int_{\alpha(t_a)}^{\alpha(t_b)}L(\bar{t},\bar{q}\circ\bar{\sigma}(\bar{t}),\bar{q}^{\bar{\Delta}}(\bar{t}))\bar{\Delta}\bar{t}$$
$$=\int_{t_a}^{t_b}L\left(\alpha(t),(\bar{q}\circ\bar{\sigma}\circ\alpha)(t),\bar{q}^{\bar{\Delta}}(\alpha(t))\right)\alpha^\Delta(t)\Delta t$$
$$=\int_{t_a}^{t_b}L\left(\alpha(t),(\bar{q}\circ\alpha\circ\sigma)(t),\frac{(\bar{q}\circ\alpha)^\Delta(t)}{\alpha^\Delta(t)}\right)\alpha^\Delta(t)\Delta t$$
$$=\int_{t_a}^{t_b}L\left(T(t,q(t),\varepsilon),Q^\sigma(t,q(t),\varepsilon),\frac{Q^\Delta}{T^\Delta}\right)T^\Delta\Delta t.$$

From the arbitrariness of t_a and t_b it follows that

$$L\left(T(t,q(t),\varepsilon),Q^\sigma(t,q(t),\varepsilon),\frac{Q^\Delta}{T^\Delta}\right)T^\Delta=L(t,q^\sigma(t),q^\Delta(t)),$$

and this implies

$$\frac{d}{d\varepsilon}\left\{L\left(T(t,q(t),\varepsilon),Q^\sigma(t,q(t),\varepsilon),\frac{Q^\Delta}{T^\Delta}\right)T^\Delta\right\}\bigg|_{\varepsilon=0}=0.$$

Lemma 1: Functional I in (1) is invariant on U under the family of transformations (19) if and only if

$$\partial_1 L(t,q^\sigma(t),q^\Delta(t))\tau(t,q(t))+\partial_2 L(t,q^\sigma(t),q^\Delta(t))\xi^\sigma(t,q(t))$$
$$+\partial_3 L(t,q^\sigma(t),q^\Delta(t))\xi^\Delta(t,q(t))+L(t,q^\sigma(t),q^\Delta(t))\tau^\Delta(t,q(t))$$
$$-q^\Delta(t)\partial_3 L(t,q^\sigma(t),q^\Delta(t))\tau^\Delta(t,q(t))=0$$

for all $t\in[a,b]_{\mathbb{T}}^\kappa$ and all $q\in U$, where

$$\xi^\sigma(t,q(t))=\xi(\sigma(t),q(\sigma(t))),\quad\xi^\Delta(t,q(t))=\frac{\Delta}{\Delta t}\xi(t,q(t)).$$

Proof: Since

$$\frac{\partial T(t,q(t),\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \tau(t,q(t)),$$

$$\frac{\partial Q^\sigma(t,q(t),\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \xi^\sigma(t,q(t)),$$

$$\frac{\partial}{\partial \varepsilon}\left(\frac{Q^\Delta(t,q(t),\varepsilon)}{T^\Delta(t,q(t),\varepsilon)}\right)\Big|_{\varepsilon=0} = \xi^\Delta(t,q(t)) - q^\Delta(t)\tau^\Delta(t,q(t)),$$

$$\frac{\partial T^\Delta(t,q(t),\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \tau^\Delta(t,q(t)),$$

the definition of invariance is equivalent to

$$\partial_1 L(t,q^\sigma(t),q^\Delta(t))\tau(t,q(t))$$
$$+ \partial_2 L(t,q^\sigma(t),q^\Delta(t))\xi^\sigma(t,q(t))$$
$$+ \partial_3 L(t,q^\sigma(t),q^\Delta(t))\left(\xi^\Delta(t,q(t)) - q^\Delta(t)\tau^\Delta(t,q(t))\right)$$
$$+ L(t,q^\sigma(t),q^\Delta(t))\tau^\Delta(t,q(t)) = 0,$$

which proves the desired result. ∎

Example 5: For Example 3 one has invariance under the family of transformations (19) with $\tau = r$ and $\xi = s$, where r and s are arbitrary constants.

In order to simplify expressions, we write $L(t,q^\sigma,q^\Delta)$ instead of $L(t,q^\sigma(t),q^\Delta(t))$. Similarly for the partial derivatives of L. We recall that q is an *extremal* to problem (1) if it satisfies the Euler-Lagrange equation (7).

Theorem 6 (Noether's theorem on time scales): If functional I in (1) is invariant on U in the sense of Definition 5 (cf. Lemma 1), then

$$\partial_3 L(t,q^\sigma,q^\Delta)\cdot\xi(t,q) + \Big[L(t,q^\sigma,q^\Delta) - \partial_3 L(t,q^\sigma,q^\Delta)\cdot q^\Delta$$
$$- \partial_1 L(t,q^\sigma,q^\Delta)\cdot\mu(t)\Big]\cdot\tau(t,q)$$

is constant along all the extremals of problem (1).

Proof: We must prove that

$$\mathscr{C} := \frac{\Delta}{\Delta t}\Big[\partial_3 L(t,q^\sigma,q^\Delta)\cdot\xi(t,q)$$
$$+ \Big(L(t,q^\sigma,q^\Delta) - \partial_3 L(t,q^\sigma,q^\Delta)\cdot q^\Delta$$
$$- \partial_1 L(t,q^\sigma,q^\Delta)\cdot\mu(t)\Big)\cdot\tau(t,q)\Big]$$

is equal to zero along all the extremals of problem (1). We begin noting that

$$\mathscr{C} = \frac{\Delta}{\Delta t}\partial_3 L(t,q^\sigma,q^\Delta)\cdot\xi^\sigma(t,q) + \partial_3 L(t,q^\sigma,q^\Delta)\cdot\xi^\Delta(t,q)$$
$$+ \frac{\Delta}{\Delta t}\Big[L(t,q^\sigma,q^\Delta) - \partial_3 L(t,q^\sigma,q^\Delta)\cdot q^\Delta$$
$$- \partial_1 L(t,q^\sigma,q^\Delta)\cdot\mu(t)\Big]\cdot\tau^\sigma(t,q)$$
$$+ \Big[L(t,q^\sigma,q^\Delta) - \partial_3 L(t,q^\sigma,q^\Delta)\cdot q^\Delta$$
$$- \partial_1 L(t,q^\sigma,q^\Delta)\cdot\mu(t)\Big]\cdot\tau^\Delta(t,q).$$

Using the first and second Euler-Lagrange equations (7) and

(8), respectively, we conclude that

$$\mathscr{C} = \partial_2 L(t,q^\sigma,q^\Delta)\cdot\xi^\sigma(t,q) + \partial_3 L(t,q^\sigma,q^\Delta)\cdot\xi^\Delta(t,q)$$
$$+ \partial_1 L(t,q^\sigma,q^\Delta)\cdot\tau^\sigma(t,q)$$
$$+ L(t,q^\sigma,q^\Delta)\cdot\tau^\Delta(t,q) - \partial_3 L(t,q^\sigma,q^\Delta)\cdot q^\Delta\cdot\tau^\Delta(t,q)$$
$$- \partial_1 L(t,q^\sigma,q^\Delta)\cdot\mu(t)\cdot\tau^\Delta(t,q).$$

Since $\tau^\sigma(t,q) = \tau(t,q) + \mu(t)\cdot\tau^\Delta(t,q)$, then

$$\mathscr{C} = \partial_2 L(t,q^\sigma,q^\Delta)\cdot\xi^\sigma(t,q) + \partial_3 L(t,q^\sigma,q^\Delta)\cdot\xi^\Delta(t,q)$$
$$+ \partial_1 L(t,q^\sigma,q^\Delta)\cdot\tau(t,q) + \partial_1 L(t,q^\sigma,q^\Delta)\cdot\mu(t)\cdot\tau^\Delta(t,q)$$
$$+ L(t,q^\sigma,q^\Delta)\cdot\tau^\Delta(t,q) - \partial_3 L(t,q^\sigma,q^\Delta)\cdot q^\Delta\cdot\tau^\Delta(t,q)$$
$$- \partial_1 L(t,q^\sigma,q^\Delta)\cdot\mu(t)\cdot\tau^\Delta(t,q).$$

Hence,

$$\mathscr{C} = \partial_2 L(t,q^\sigma,q^\Delta)\cdot\xi^\sigma(t,q) + \partial_3 L(t,q^\sigma,q^\Delta)\cdot\xi^\Delta(t,q)$$
$$+ \partial_1 L(t,q^\sigma,q^\Delta)\cdot\tau(t,q) + L(t,q^\sigma,q^\Delta)\cdot\tau^\Delta(t,q)$$
$$- \partial_3 L(t,q^\sigma,q^\Delta)\cdot q^\Delta\cdot\tau^\Delta(t,q).$$

Using Lemma 1 we arrive at the intended conclusion. ∎

If $\mathbb{T} = \mathbb{R}$, then $\mu = 0$ and Theorem 6 reduces to the classical Noether's theorem (cf., e.g., [27]):

Corollary 3 (Noether's theorem): If the classical fundamental functional of the calculus of variations (2) is invariant, then

$$\partial_3 L(t,q,\dot{q})\cdot\xi(t,q) + [L(t,q,\dot{q}) - \partial_3 L(t,q,\dot{q})\cdot\dot{q}]\cdot\tau(t,q)$$

is constant along all the extremals of the problem.

Example 6: For the problem of Example 5 one has from Theorem 6 that

$$2sq^\Delta - r(q^\Delta)^2 = const \qquad (20)$$

along the extremals q of the problem. This is indeed true: from Example 3 we know that the extremals have the form $q(t) = ct + k$ for some constants $c,k \in \mathbb{R}$; thus, the conservation law (20) takes the form $2sc - rc^2 = const$.

V. CONCLUSION AND FUTURE WORK

In this paper we obtain a second Euler-Lagrange equation and a second Erdmann condition for the problem of the calculus of variations on time scales. Since both necessary optimality conditions are important and extremely useful results in the calculus of variations and optimal control when $\mathbb{T} = \mathbb{R}$, we claim that the present results are also useful for the development of the recent theory of the calculus of variations on time scales [19]. As pointed out to us by Richard Vinter, our second Euler-Lagrange equation in the time scales setting seems to be useful in a framework for studying the asymptotics of time discretization.

As an example of application of our main results, we give a simpler and more elegant proof to the Noether symmetry theorem on time scales obtained in 2008 [11], which

allows to obtain conserved quantities along the extremals of the problems. Standard Noetherian constants of motion are violated due to the presence of a new term that depends on the graininess $\mu(t)$ of the time scale, while in the classical context $\mu(t) \equiv 0$. The importance of Noether's conservation laws in the calculus of variations, optimal control theory, and its applications in engineering, are well recognized [12], [21], [22], [33]. Their role on the general context of optimal control on time scales is an entirely open area of research. In particular, it would be interesting to investigate the techniques of [33], [36] with the recent higher-order Euler-Lagrange equations on time scales [20], [30] for a possible extension of Theorem 6 to variational problems on time scales with higher-order delta or nabla derivatives. The question of obtaining conserved quantities along the extremals of higher-order problems of the calculus of variations on time scales remains an interesting open question.

VI. ACKNOWLEDGMENTS

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